Branched Covers
Gabriel Ong Prof. Eric Ramos
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Branched covers are generalizations of covering spaces, and are widely used in geometry and topology. Herein, we will attempt to give a brief overview of the subject following John Etnyre's wonderfully illustrated notes [1] leading to a proof of the Riemann-Hurwitz theorem and of Hurwitz' theorem. Let us recall the definition.

Definition. *Let X, Y be n-manifolds and $f : X \to Y$ such that $f^{-1}(\partial Y) = \partial X$. $f : X \to Y$ is a d-fold branched covering if there is a CW subcomplex $B \subseteq Y$ of codimension 2, the branch locus, such that $f^{-1}(B) \subseteq X$ is also codimension 2 and $f|_{Y \setminus f^{-1}(B)} : X \setminus f^{-1}(B) \to Y \setminus B$ is a d-sheeted covering map.*

Remark. *The first condition tells us that the branch locus is "small" with respect to the base space, in particular, it has dimension 2 less than the base space (eg. the branch locus of a 2-manifold would have dimension 0 – branch points).*

We can thus think of a branched cover as a d-sheeted cover away from the branch locus. An easy example comes from elementary complex analysis.

Example. *$f : \mathbb{C} \to \mathbb{C}$ by $z \mapsto z^k$ is a k-fold branched covering with branch locus $\{0\}$.*

Herein, we will consider branched covers of surfaces, ie. covers of 2-manifolds that are branched over points. These points will have neighborhood $z \mapsto z^k$ for $k \in \mathbb{N}$. A point which is locally $z \mapsto z$ is a simple point, while a point which is locally $z \mapsto z^k$ for $k \in \mathbb{N}_{\geq 2}$ has branch index k. A branched cover is simple if all points are locally $z \mapsto z^k$ for $k \in \{1, 2\}$. We can now state and prove the Riemann-Hurwitz theorem.

Theorem (Riemann-Hurwitz)**.** *If $f : X \to Y$ is a d-fold branched cover of surfaces branched along $B = \{y_1, \ldots, y_k\} \subseteq Y$ with $f^{-1}(B) = \{x_1, \ldots, x_n\} \subseteq X$ where $a_i = |f^{-1}(y_i)|$ and b_j the branch index of x_i then*

$$\chi(X) = d(\chi(Y) - k) + \sum_{i=1}^{k} a_i = d\chi(Y) - \sum_{j=1}^{n} (b_j - 1).$$

Proof. Without loss of generality, endow Y with a CW structure where B a collection of 0-cells. So $Y \setminus B$ is a CW complex with the same number of 1-cells and 2-cells but k less 0-cells than Y and thus $\chi(Y \setminus B) = \chi(Y) - k$. Moreover, with the hypothesis that

Now note that $f^{-1}(Y \setminus B) = X \setminus f^{-1}(B)$ is a CW substructure of X with $\sum_{i=1}^{k} a_i$ 0-cells removed. So $\chi(X \setminus f^{-1}(B)) = \chi(X) - \sum_{i=1}^{k} a_i = d(\chi(Y) - k)$ where the final equality is from [2, Ex. 2.2.22]. Rearranging, we yield $\chi(X) = d(\chi(Y) - k) + \sum_{i=1}^{k} a_i$ proving the first equality.

Finally, expanding the formula above, $\chi(X) = d\chi(Y) - dk + \sum_{i=1}^{k} a_i$ so to show the second equality it suffices to show $-dk + \sum_{i=1}^{k} a_i = -\sum_{j=1}^{n} (b_j - 1) = n - \sum_{j=1}^{n} b_j$. For this, observe that the fiber over each branch point of the base y_i is a collection of finitely many x_j whose branching indices sum to d. Summing over the k fibers, we yield $\sum_{j=1}^{n} b_j = dk$ and thus we want to show $-dk + \sum_{i=1}^{k} a_k = n - dk$. But indeed $\sum_{i=1}^{k} a_k = |f^{-1}(B)| = n$ demonstrating the desired equality and completing the proof of the theorem. □

The Riemann-Hurwitz theorem places conditions on the space X. We can either fix the fold of the cover or the number of branch points, but not both. This leads to the following corollaries.

Corollary. *Let Σ_g be a genus g topological surface. Σ_g is a 2-sheeted branched cover of S^2 branched at $2g + 2$ points.*

Note that the cover is necessarily simply branched since the preimage of any point of the base is either 2 or 1.

Proof. Here $d = 2$, $\chi(\Sigma_g) = 2 - 2g$, and $\chi(S^2) = 2$. Moreover each preimage $a_i = 1$ since the preimage of a branch point is strictly less than 2 – the number of sheets – but nonempty. So $2 - 2g = 2(2 - k) + k$ and thus $k = 2g + 2$. □

Corollary. *Let Σ_g be a genus g topological surface. Σ_g is a d-sheeted branched cover at S^2 branched at 3 points where $d = 2g - 2 + \sum_{i=1}^{k} a_i$.*

Proof. Here $\chi(\Sigma_g) = 2 - 2g$ and $\chi(S^2) = 2$. So $2 - 2g = d(2-3) + \sum_{i=1}^{k} a_k$ and thus $d = 2g - 2 + \sum_{i=1}^{k} a_k$. □

This shows us that for any genus g surface Σ_g as a cover of S^2 branched at three points, its "sheetedness" as a cover is determined by the cardinality of the preimages at the branch points.

One particularly nice class of branched covers are simply branched covers, those where $|f^{-1}(y)| \in \{d, d-1\}$ for all $y \in Y$. Around the turn of the 20th Century, Hurwitz proposed the following problem:

Question. *For given d, k, what is the Hurwitz number $\mathcal{H}_{d,k}$, the number of simply branched d-sheeted covers of S^2 branched at $k = 2d + 2g - 2$ points?*

He eventually proved the following:

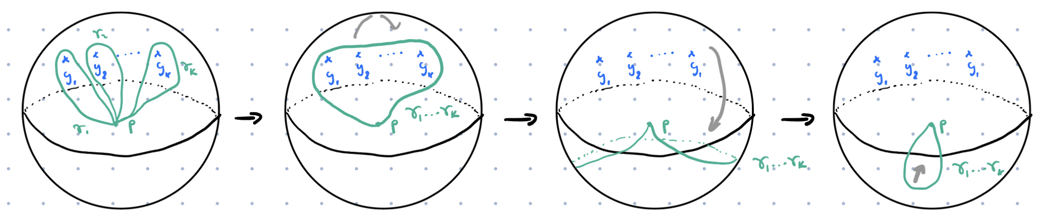
Theorem (Hurwitz)**.** *For given d, k, the Hurwitz number $\mathcal{H}_{d,k}$ is $\frac{1}{d!}N$ where N is the number of tuples $(\tau_1, \dots, \tau_k) \in (S_d)^{\oplus k}$ satisfying*

$$\tau_1 \dots \tau_k = \epsilon \in S_d$$

where $\tau_i^2 = \epsilon$ for all i.

It is perhaps not surprising that the theory of the symmetric group arises here. Branched covers, as generalizations of covering spaces, enjoy similar properties. In particular, they are determined by their monodromy. It thus suffices to determine all possible monodromies of the cover. The exposition of this proof follows that of [3].

Outline of Proof of Hurwitz' Theorem. Let $f : X \to S^2$ be a branched cover of the sphere along $k = 2d + 2g - 2$ points. Pick some $p \in S^2 \setminus B$ and consider loops $\gamma_1, \dots, \gamma_k$ around each of the branch points $y_1, \dots, y_k \in S^2$ where the product $\gamma_1 \dots \gamma_k = \text{id} \in \pi_1(S^2 \setminus B, p)$ as shown in the following figure.



Each loop $\gamma \in \pi_1(S^2 \setminus B, p)$ lifts to a path $f^{-1}\gamma$ between points in the fiber over p. Since the cover is simply branched, the lift of each loop γ_i is a path $f^{-1}\gamma_i$ which induces a monodromy that is a transposition in S_d that we denote $\tau_i \in S_d$. The lift of ϵ has induced monodromy ϵ so the τ_i are subject to the condition that $\tau_1 \dots \tau_k = \epsilon$ and thus the answer to our problem is the number of k-tuples of transpositions in S_d whose product is the identity, N. However, this count is up to a choice of labeling of the sheets $f^{-1}(p)$, of which there are $d!$ ways to do so. Thus dividing N by $d!$ yields the correct solution. □

References

[1] John Etnyre. *Branched Covers.* URL: `https://etnyre.math.gatech.edu/Branched%20Coverings.pdf`.

[2] Allen Hatcher. *Algebraic Topology.* Cambridge University Press, 2002.

[3] Rahul Pandharipande. *Algebraic Curves, Hurwitz Covers and Meromorphic Differentials.* 2021. URL: `https://youtu.be/_DMpiMaZ9Vs`.